Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | Candidate Number

Pearson Edexcel International Advanced Level

Tuesday 30 May 2023

Afternoon (Time: 1 hour 30 minutes)

Paper reference **WFM01/01**

Mathematics

International Advanced Subsidiary/Advanced Level
Further Pure Mathematics F1

You must have:
Mathematical Formulae and Statistical Tables (Yellow), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over

P 7 3 4 8 6 A 0 1 3 2

Pearson

1. Use the standard results for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$ to show that, for all positive integers n

$$\sum_{r=1}^{n} r^2(r+2) = \frac{1}{12}n(n+1)(an^2+bn+c)$$

where a, b and c are integers to be determined.

(4)

$\sum_{r=1}^{n} r^2(r+2) = \sum_{r=1}^{n} r^3 + 2r^2$

can split summation into 2

$\sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} 2r^2$

factor 2 out summation

$= \sum_{r=1}^{n} r^3 + 2\sum_{r=1}^{n} r^2$

$= \frac{1}{4}n^2(n+1)^2 + 2\left[\frac{1}{6}n(n+1)(2n+1)\right]$

$= \frac{1}{4}n^2(n+1)^2 + \frac{2}{6}n(n+1)(2n+1)$

Summations

$\sum_{r=1}^{n} 1 = n$ must learn!

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ in the formulae booklet

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{3}n(n+1)(2n+1)$

factor out $\frac{1}{12}n(n+1)$

$= \frac{1}{12}n(n+1)\left[3n(n+1) + 4(2n+1)\right]$

$= \frac{1}{12}n(n+1)\left[3n^2+3n+8n+4\right]$

$= \frac{1}{12}n(n+1)\left[3n^2+11n+4\right]$

$\boxed{\frac{1}{12}n(n+1)(3n^2+11n+4) \quad a=3, b=11, c=4}$

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

P 7 3 4 8 6 A 0 2 3 2

2. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Given that $x = 2 + 3i$ is a root of the equation

$$2x^4 - 8x^3 + 29x^2 - 12x + 39 = 0$$

(a) write down another complex root of this equation.

(1)

(b) Use algebra to determine the other 2 roots of the equation.

(4)

(c) Show all 4 roots on a single Argand diagram.

(2)

a. For a quartic eqⁿ with form: $ax^4 + bx^3 + cx^2 + dx + e$, there are 4 roots.
There are 3 possibilities: 4 real roots
2 real roots and 2 complex roots (conjugate pair)
4 complex roots (2 conjugate pairs).

The other complex root must be the complex conjugate of $(2+3i)$
(flip sign of imaginary component

$\boxed{2-3i}$

b. $(x-(2-3i))(x-(2+3i))(ax^2+bx+c)$

$(x-2+3i)(x-2-3i)(ax^2+bx+c)$

$(x^2-2x-\cancel{3ix}-2x+4+\cancel{6i}+\cancel{3ix}-\cancel{6i}-9i^2)(ax^2+bx+c)$

$(x^2-4x+4-9(-1))(ax^2+bx+c)$

$(x^2-4x+13)(ax^2+bx+c)$

$ax^4+bx^3+cx^2-4ax^3-4bx^2-4cx+13ax^2+13bx+13c$

$(a)x^4+(b-4a)x^3+(c-4b+13a)x+(-4c+13b)x+(13c)$

Compare coefficents with eqⁿ above:
(1) $a=2$
(2) $b-4a=-8$
(3) $c-4b+13a=29$
(4) $-4c+13b=-12$
(5) $13c=39$

based off of (1) and (5), $a=2$ and $13c=39$

$c=\frac{39}{13}$

$c=3$

P 7 3 4 8 6 A 0 4 3 2

Question 2 continued

Sub $a=2$ into (2) and solve for b.

$b-4a=-8$

$b-4(2)=-8$

$b-8=-8$

$b=0$

$\therefore$ quadratic factor is $2x^2+3$

& find roots

$2x^2+3=0$

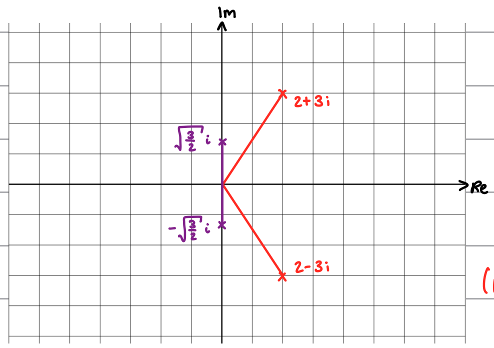
$2x^2=-3$

$x^2=-\frac{3}{2}$

$x=\pm\sqrt{-\frac{3}{2}}=\pm\sqrt{\frac{3}{2}}i$

$\boxed{x=+\sqrt{\frac{3}{2}}i,-\sqrt{\frac{3}{2}}i}$

c.



(NOT DRAWN TO SCALE)

3. The rectangular hyperbola H has Cartesian equation $xy = 9$

The point P with coordinates $\left(3t, \frac{3}{t}\right)$, where $t \neq 0$, lies on H

(a) Use calculus to determine an equation for the normal to H at the point P

Give your answer in the form $ty - t^3x = \mathrm{f}(t)$

(4)

Given that $t = 2$

(b) determine the coordinates of the point where the normal meets H again.

Give your answer in simplest form.

(3)

a. Differentiate rectangular hyperbola eq^n w.r.t. x:

$xy = 9$

$y = \frac{9}{x} = 9x^{-1}$

$\frac{dy}{dx} = -9x^{-2} = \frac{-9}{x^2}$

Sub in point $P\left(3t, \frac{3}{t}\right)$:

$\left.\frac{dy}{dx}\right|_{(3t, \frac{3}{t})} \quad \frac{-9}{(3t)^2} = \frac{-9}{9t^2} = \frac{-1}{t^2}$

$M_{tangent} : -\frac{1}{t^2}$

$M_{normal} : t^2$

Set up line eq^n $y - y_1 = m(x - x_1)$ with $m = t^2$

$(x_1, y_1) = \left(3t, \frac{3}{t}\right)$

$y - \frac{3}{t} = t^2(x - 3t)$

$y - \frac{3}{t} = t^2x - 3t^3$ $\quad \times t$

$ty - 3 = t^3x - 3t^4$

$ty - t^3x = 3 - 3t^4$

$\boxed{ty - t^3x = 3 - 3t^4 \quad f(x) = 3 - 3t^4}$

b. Sub $t = 2$ into normal line eq^n:

$(2)y - (2)^3x = 3 - 3(2)^4$

$2y - 8x = -45$

Question 3 continued

Solve normal line eqⁿ and rectangular hyperbola eqⁿ H simultaneously.

(1) $2y - 8x = -45$

(2) $xy = 9$

(2) $xy = 9$ $\times 2$

$2xy = 18$

$2y = \frac{18}{x}$

(1) $2y - 8x = -45$

$\left(\frac{18}{x}\right) - 8x = -45$

$18 - 8x^2 = -45x$

$8x^2 - 45x - 18 = 0$

$(8x+3)(x-6) = 0$

$x = -\frac{3}{8}$ or $x = 6$

This is coord. $P\left(3t, \frac{3}{t}\right)\Big|_{t=2} = P\left(6, \frac{3}{2}\right)$

Sub into either (1) or (2) and solve for y

$xy = 9$

$y = \frac{9}{x}$

$y = \frac{9}{(-3/8)} = -24$

$\boxed{\left(-\frac{3}{8}, -24\right)}$

(Total for Question 3 is 7 marks)

DO NOT WRITE IN THIS AREA

4. (i) $\mathbf{A} = \begin{pmatrix} -3 & 8 \\ -3 & k \end{pmatrix}$ where k is a constant

The transformation represented by **A** transforms triangle T to triangle T'

The area of triangle T' is three times the area of triangle T

Determine the possible values of k

(4)

(ii) $\mathbf{B} = \begin{pmatrix} a & -4 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{BC} = \begin{pmatrix} 2 & 5 & 1 \\ 1 & 4 & 2 \end{pmatrix}$ where a is a constant

Determine, in terms of a, the matrix **C**

(4)

i. $|\det(M)|$ = area scale factor

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(X) = (a)(d) - (b)(c)$

$\det(A) = (-3)(k) - (8)(-3)$

$= -3k + 24$

$\det(A) = -3k + 24$

Area scale factor $= |\det(A)| = |-3k + 24|$

Area of T × 3 = Area of T'

$3 = |-3k + 24|$

$3 = -3k + 24$ or $3 = -(-3k + 24)$

$3k = 21$ $3 = 3k - 24$

$k = 7$ $3k = 27$

$k = 9$

$\boxed{k = 7 \text{ or } k = 9}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 4 continued

ii. $B \times C = (BC)$

$\cancel{B^{-1}B}\, C = B^{-1}(BC)$

$\therefore C = B^{-1}(BC)$

(1) find B^{-1} first

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(x) = (a)(d) - (b)(c)$

$\det(B) = (a)(3) - (2)(-4)$

$= 3a + 8$

$\det(B) = 3a + 8$

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X^{-1} = \frac{1}{\det(x)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$-b = -(-4) = 4$

$-c = -(2) = -2$

$B^{-1} = \frac{1}{3a+8} \begin{bmatrix} 3 & 4 \\ -2 & a \end{bmatrix}$

(2) sub B^{-1} matrix into $C = B^{-1}(BC)$

$C = B^{-1}(BC)$

$C = \frac{1}{3a+8} \begin{bmatrix} 3 & 4 \\ -2 & a \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 \\ 1 & 4 & 2 \end{bmatrix}$

$C = \frac{1}{3a+8} \begin{bmatrix} (3)(2)+(4)(1) & (3)(5)+(4)(4) & (3)(1)+(4)(2) \\ (-2)(2)+(a)(1) & (-2)(5)+(a)(4) & (-2)(1)+(a)(2) \end{bmatrix}$

$\boxed{C = \frac{1}{3a+8} \begin{bmatrix} 10 & 31 & 11 \\ a-4 & 4a-10 & 2a-2 \end{bmatrix}}$

5.
$$f(x) = x^2 - 6x + 3$$

The equation $f(x) = 0$ has roots α and β

Without solving the equation,

(a) determine the value of

$$(\alpha^2 + 1)(\beta^2 + 1)$$

(4)

(b) find a quadratic equation which has roots

$$\frac{\alpha}{(\alpha^2 + 1)} \text{ and } \frac{\beta}{(\beta^2 + 1)}$$

giving your answer in the form $px^2 + qx + r = 0$ where p, q and r are integers to be determined.

(6)

a. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

let α, β be roots of quadratic

$x^2 - (\alpha + \beta)x + (\alpha\beta) = 0$

sum of roots : $\alpha + \beta$

product of roots: $\alpha\beta$

$x^2 - 6x + 3 = 0$

$\alpha + \beta = 6$

$\alpha\beta = 3$

$(\alpha^2 + 1)(\beta^2 + 1) = \alpha^2\beta^2 + \alpha^2 + \beta^2 + 1$

$= (\alpha\beta)^2 + (\alpha + \beta)^2 - 2\alpha\beta + 1$

$= (3)^2 + (6)^2 - 2(3) + 1$

$= 40$

$\boxed{(\alpha^2 + 1)(\beta^2 + 1) = 40}$

Question 5 continued

b. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$$x^2 - \left(\frac{\alpha}{\alpha^2+1} + \frac{\beta}{\beta^2+1}\right)x + \left(\left(\frac{\alpha}{\alpha^2+1}\right) \times \left(\frac{\beta}{\beta^2+1}\right)\right) = 0$$

$$x^2 - \left(\frac{\alpha(\beta^2+1) + \beta(\alpha^2+1)}{(\alpha^2+1)(\beta^2+1)}\right)x + \left(\frac{\alpha\beta}{(\alpha^2+1)(\beta^2+1)}\right) = 0$$

use ans to part (a)

$$x^2 - \left(\frac{\alpha\beta^2 + \alpha + \beta\alpha^2 + \beta}{(\alpha^2+1)(\beta^2+1)}\right)x + \left(\frac{\alpha\beta}{(\alpha^2+1)(\beta^2+1)}\right) = 0$$

$$x^2 - \left(\frac{\alpha\beta^2 + \beta\alpha^2 + \alpha + \beta}{(\alpha^2+1)(\beta^2+1)}\right)x + \left(\frac{\alpha\beta}{(\alpha^2+1)(\beta^2+1)}\right) = 0$$

$$x^2 - \left(\frac{\alpha\beta(\beta+\alpha) + \alpha + \beta}{(\alpha^2+1)(\beta^2+1)}\right)x + \left(\frac{\alpha\beta}{(\alpha^2+1)(\beta^2+1)}\right) = 0$$

$$x^2 - \left(\frac{(3)(6) + (6)}{(40)}\right)x + \left(\frac{(3)}{(40)}\right) = 0$$

$$x^2 - \left(\frac{3}{5}\right)x + \left(\frac{3}{40}\right) = 0$$

$$x^2 - \frac{3}{5}x + \frac{3}{40} = 0$$

$$x^2 - \frac{3}{5}x + \frac{3}{40} = 0 \quad \times 40$$

$$40x^2 - 24x + 3 = 0 \quad \longleftarrow \text{all integer coefficents}$$

$$\boxed{40x^2 - 24x + 3 = 0 \qquad p = 40,\ q = -24,\ r = 3}$$

6. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

$$z_1 = 3 + 2\mathrm{i} \qquad z_2 = 2 + 3\mathrm{i} \qquad z_3 = a + b\mathrm{i} \qquad a,b \in \mathbb{R}$$

(a) Determine the exact value of $|z_1 + z_2|$

(2)

Given that $w = \dfrac{z_2 z_3}{z_1}$

(b) determine w in terms of a and b, giving your answer in the form $x + \mathrm{i}y$, where $x, y \in \mathbb{R}$

(4)

Given also that $w = \dfrac{4}{13} + \dfrac{58}{13}\mathrm{i}$

(c) determine the value of a and the value of b

(2)

(d) determine arg w, giving your answer in radians to 4 significant figures.

(2)

a. $|z_1 + z_2|$

$|(3+2i) + (2+3i)|$

$|5+5i|$

$\sqrt{(5)^2 + (5)^2} = 5\sqrt{2}$

$\boxed{|z_1 + z_2| = 5\sqrt{2}}$

b. $W = \dfrac{z_2 z_3}{z_1} = \left(\dfrac{z_2}{z_1}\right) \times z_3$ work out $\left(\frac{z_2}{z_1}\right)$ first, then multiply by z_3 after

$= \left(\dfrac{(2+3i)}{(3+2i)}\right) \times (a+bi)$

multiply top and bottom by complex conjugate of denominator

$\dfrac{2+3i}{3+2i} \times \dfrac{3-2i}{3-2i} = \dfrac{(2+3i)(3-2i)}{(3+2i)(3-2i)}$

$= \dfrac{6-4i+9i-6i^2}{9-6i+6i-4i^2}$

$= \dfrac{6-6(-1)+5i}{9-4(-1)}$

P 7 3 4 8 6 A 0 1 8 3 2

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 6 continued

$$\frac{12+5i}{13}$$

$$= \frac{12}{13} + \frac{5}{13}i$$

$$\therefore \left(\frac{2+3i}{3+2i}\right) = \frac{12}{13} + \frac{5}{13}i$$

$$\left(\frac{12}{13} + \frac{5}{13}i\right) \times (a+bi)$$

$$= \frac{12}{13}a + \frac{12}{13}bi + \frac{5}{13}ai + \frac{5}{13}bi^2$$

$$= \frac{12}{13}a + \frac{5}{13}b(-1) + \left(\frac{5}{13}a + \frac{12}{13}b\right)i$$

$$= \left(\frac{12}{13}a - \frac{5}{13}b\right) + \left(\frac{5}{13}a + \frac{12}{13}b\right)i$$

$$\boxed{W = \left(\frac{12a-5b}{13}\right) + \left(\frac{5a+12b}{13}\right)i \qquad x = \left(\frac{12a-5b}{13}\right), \quad y = \left(\frac{5a+12b}{13}\right)}$$

c. $\left(\frac{12a-5b}{13}\right) + \left(\frac{5a+12b}{13}\right)i = \frac{4}{13} + \frac{58}{13}i$

Equate the real components and imaginary components:

$$\frac{12a-5b}{13} = \frac{4}{13} \implies 12a - 5b = 4 \quad (1)$$

$$\frac{5a+12b}{13} = \frac{58}{13} \implies 5a + 12b = 58 \quad (2)$$

Question 6 continued

Solve (1) and (2) simultaneously:

(1) $12a - 5b = 4$

(2) $5a + 12b = 58$

(1) × 12 $\quad 144a - 60b = 48$

(2) × 5 $\quad 25a + 60b = 290$ (+)

$169a = 338$

$169a = 338$

$a = \frac{338}{169}$

$a = 2$

$12a - 5b = 4$

$12(2) - 5b = 4$

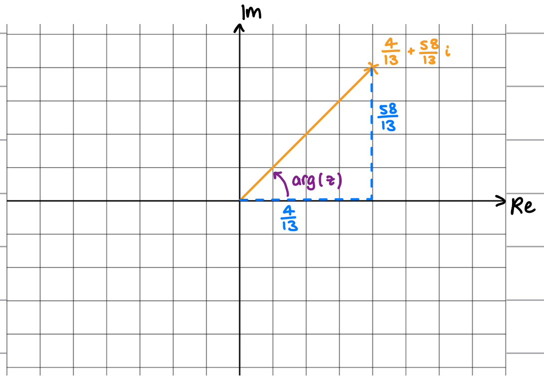
$24 - 5b = 4$

$20 = 5b$

$b = 4$

$\boxed{a = 2,\ b = 4}$

d.



$\arg(w) = \tan^{-1}\left(\frac{58/13}{4/13}\right)$

$\arg(w) = 1.501939837^c$

$\boxed{\arg(w) = 1.502^c \ (4\text{s.f.})}$

7.

$$f(x) = x^{\frac{3}{2}} + x - 3$$

(a) Show that the equation $f(x) = 0$ has a root, α, in the interval [1, 2]

(2)

(b) Starting with the interval [1, 2], use interval bisection twice to show that α lies in the interval [1.25, 1.5]

(3)

(c) (i) Determine $f'(x)$

(ii) Using 1.375 as a first approximation for α, apply the Newton-Raphson process once to $f(x)$ to determine a second approximation for α, giving your answer to 3 decimal places.

(3)

(d) Use linear interpolation once on the interval [1.25, 1.5] to obtain a different approximation for α, giving your answer to 3 decimal places.

(3)

a. $f(1) = (1)^{\frac{3}{2}} + (1) - 3 = -1$

$f(2) = (2)^{\frac{3}{2}} + (2) - 3 = 1.828427125$

$\boxed{f(x) \text{ is continuous and changes sign so, } \alpha \text{, lies in } [1,2]}$

b. 1st interval bisection : $\frac{1+2}{2} = 1.5$

(-) (+)
1 1.5 2

$f(1.5) = (1.5)^{3/2} + (1.5) - 3 = 0.337117307$

2nd interval bisection : $\frac{1+1.5}{2} = 1.25$

(-) (+)
1 1.25 1.5

$f(1.25) = (1.25)^{3/2} + (1.25) - 3 = -0.3524575141$

(-) (+)
1.25 1.5

$\boxed{1.25 \leq \alpha \leq 1.75 \text{ // shown}}$

Question 7 continued

ci. $f(x) = x^{\frac{3}{2}} + x - 3$

$f'(x) = \frac{3}{2}x^{\frac{1}{2}} + 1$

$\boxed{f'(x) = \frac{3}{2}\sqrt{x} + 1}$

ii. $X_1 = 1.75$

we are looking for X_2

Numerical Sol^{n}s of Eq^{n}s:

Newton - Rhapson iteration for solving $f(x) = 0$

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$$

From Newton- Rhapson iteration: $X_2 = X_1 - \frac{f(X_1)}{f'(X_1)}$

(1) must find $f(X_1)$ by subbing in X_1 (1.75) into $f(x)$

(2) must find $f'(X_1)$ by differentiating $f(x)$ and subbing in X_1(1.75) into $f'(x)$.

(1) $f(1.375) = (1.375)^{\frac{3}{2}} + (1.375) - 3 = -0.01266958256$

(2) $f'(x) = \frac{3}{2}\sqrt{x} + 1$

$f'(1.375) = \frac{3}{2}\sqrt{1.375} + 1 = 2.75890591$

$$X_2 = (1.375) - \frac{-0.01266958256}{2.75890591}$$

$X_2 = 1.379592249$

$\boxed{\alpha = 1.380 \ (3 \text{ d.p.})}$

Question 7 continued

a: Linear Interpolation:

$\alpha = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ for interval $[a,b]$

where α is the root

$a = 1.25 \quad b = 1.5$

$\alpha = \dfrac{1.25 f(1.5) - 1.5 f(1.25)}{f(1.5) - f(1.25)}$

$f(1.25) = (1.25)^{3/2} + (1.25) - 3 = -0.352457514$

$f(1.5) = (1.5)^{3/2} + (1.5) - 3 = 0.3371173071$

$\alpha = \dfrac{1.25(0.3371173071) - 1.5(-0.352457514)}{0.3371173071 - -0.352457514}$

$\alpha = 1.377786731$

$\boxed{\alpha = 1.378 \ (3\text{d.p.})}$

8. The point $P(2p^2, 4p)$ lies on the parabola with equation $y^2 = 8x$

(a) Show that the point $Q\left(\dfrac{2}{p^2}, \dfrac{-4}{p}\right)$, where $p \neq 0$, lies on the parabola.

(1)

(b) Show that the chord PQ passes through the focus of the parabola.

(4)

The tangent to the parabola at P and the tangent to the parabola at Q meet at the point R

(c) Determine, in simplest form, the coordinates of R

(8)

a. To show that point Q lies on parabola, sub in $y = -\frac{4}{p}$ into $y^2 = 8x$ and should get $x = \frac{2}{p}$

$y^2 = 8x$

$\left(-\frac{4}{p}\right)^2 = 8x$

$\frac{16}{p^2} = 8x$

$\frac{2}{p^2} = x$ // shown.

$\boxed{\therefore \text{ Point Q lies on parabola}_{/\!/}}$

b. $y^2 = 8x$

$y^2 = 4(2)x \Rightarrow a = 2$

Focus : $(a, 0)$

$\therefore$ Focus of parabola : $(2, 0)$

Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e < 1$ $b^2 = a^2(1 - e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2 - 1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x = \pm\frac{a}{e}$	$x = -a$	$x = \pm\frac{a}{e}$	$x + y = \pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a} = \pm\frac{y}{b}$	$x = 0, y = 0$

Find line eq^n of chord PQ:

(1) find gradient (M_{PQ})

Gradient : $\dfrac{y_2 - y_1}{x_2 - x_1}$ for (x_1, y_1) and (x_2, y_2)

$$M_{PQ} = \frac{(4p) - \left(-\frac{4}{p}\right)}{(2p^2) - \left(\frac{2}{p^2}\right)} = \frac{4p + \frac{4}{p}}{2p^2 - \frac{2}{p^2}} \frac{\times p^2}{\times p^2} = \frac{4p^3 + 4p}{2p^4 - 2} = \frac{\cancel{2}(2p^3 + 2p)}{\cancel{2}(p^4 - 1)} = \frac{2p\cancel{(p^2+1)}}{\cancel{(p^2+1)}(p^2-1)} = \frac{2p}{p^2 - 1}$$

(2) set up line eq^n $y - y_1 = m(x - x_1)$ with $M_{PQ} : \frac{2p}{p^2 + 1}$

$(x_1, y_1) : (2p^2, 4p)$ ← can pick either point P or Q but we will pick P for simplicity

$y - 4p = \frac{2p}{p^2 - 1}(x - 2p^2)$

$y - 4p = \frac{2p}{p^2 - 1}x - \frac{4p^3}{p^2 - 1}$

P 7 3 4 8 6 A 0 2 6 3 2

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 8 continued

$$y - 4p = \frac{2p}{p^2-1}x - \frac{4p^3}{p^2-1}$$

$$y = \frac{2p}{p^2+1}x + \left(4p - \frac{4p^3}{p^2-1}\right)$$

$$y = \frac{2p}{p^2+1}x + \left(\frac{4p(p^2-1) - 4p^3}{p^2-1}\right)$$

$$y = \frac{2p}{p^2+1}x + \left(\frac{\cancel{4p^3} - 4p - \cancel{4p^3}}{p^2-1}\right)$$

$$y = \frac{2p}{p^2+1}x - \frac{4p}{p^2-1} \quad // \quad \text{line eq}^n \text{ PQ}$$

Sub in $x = 2$, $y = 0$

$$(0) - 4p = \frac{2p}{p^2+1}(2) - \frac{4p^2}{p^2+1}$$

$$-4p = \frac{4p}{p^2+1} - \frac{4p^2}{p^2+1}$$

$$-4p = \frac{4p - 4p^2}{p^2+1}$$

$$-4p = \frac{-4p\cancel{(1+p^2)}}{\cancel{p^2+1}}$$

$$-4p = -4p \qquad \text{LHS} = \text{RHS}$$

$\boxed{\therefore \text{ chord PQ passes through focus of parabola}}$

c. (1) Find eq^n of tangent @ P:

Differentiate parabola eq^n w.r.t. x:

$$y^2 = 8x$$

$$2y\left(\frac{dy}{dx}\right) = 8$$

$$\frac{dy}{dx} = \frac{8}{2y} = \frac{4}{y}$$

Sub in point $P(2p^2, 4p)$:

$$\left.\frac{dy}{dx}\right|_{(2p^2,4p)} \quad \frac{4}{4p} = \frac{1}{p}$$

$$m_{\text{tangent}} = \frac{1}{p}$$

Set up line eq^n $y - y_1 = m(x - x_1)$ with $m = \frac{1}{p}$

$(x_1, y_1) = (2p^2, 4p)$

$$y - 4p = \frac{1}{p}(x - 2p^2)$$

$$y - 4p = \frac{1}{p}x - 2p$$

$$y = \frac{1}{p}x + 2p \quad ☆$$

Question 8 continued

(2) Find eq^n of tangent @ Q:

Differentiate parabola eq^n w.r.t. x:

$y^2 = 8x$

$2y\left(\frac{dy}{dx}\right) = 8$

$\frac{dy}{dx} = \frac{8}{2y} = \frac{4}{y}$

Sub in point $Q\left(\frac{2}{p^2}, -\frac{4}{p}\right)$:

$\left.\frac{dy}{dx}\right|_{\left(\frac{2}{p^2}, -\frac{4}{p}\right)} \quad \frac{4}{\left(-\frac{4}{p}\right)} = -p$

$m_{\text{tangent}} = -p$

Set up line eq^n $y - y_1 = m(x - x_1)$ with $m = -p$

$(x_1, y_1) = \left(\frac{2}{p^2}, -\frac{4}{p}\right)$

$y - -\frac{4}{p} = -p\left(x - \frac{2}{p^2}\right)$

$y + \frac{4}{p} = -px + \frac{2}{p}$

$y = -px - \frac{2}{p}$ ☆

Solve (1) and (2) simultaneously to find coord. R:

(1) $y = \frac{1}{p}x + 2p$

(2) $y = -px - \frac{2}{p}$

$\frac{1}{p}x + 2p = -px - \frac{2}{p}$ $\times 2$

$x + 2p^2 = -p^2x - 2$

$x + p^2x = -2p^2 - 2$

$x(1 + p^2) = -2(p^2 + 1)$

$x = \frac{-2\cancel{(p^2+1)}}{\cancel{p^2+1}} = -2$

Sub $x = -2$ into either (1) or (2) to find y

$y = \frac{1}{p}(-2) + 2p = \frac{-2}{p} + 2p$

$\boxed{R\left(-2, -\frac{2}{p} + 2p\right)}$

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

P 7 3 4 8 6 A 0 2 8 3 2

9. Prove, by induction, that for $n \in \mathbb{Z}, n \geqslant 2$

$$4^n + 6n - 10$$

is divisible by 18

(5)

(1) Base Case: check true for $n=2$:

$f(2) = 4^2 + 6(2) - 10$

$= 18$

$= 1(18) \leftarrow$ divisible by 18

$\therefore$ true for $n=2$

(2) Assume true for $n=k$:

$f(k) = 4^k + 6k - 10$

(3) Inductive step, prove true for $n=k+1$:

$f(k+1) = 4^{k+1} + 6(k+1) - 10$

$= 4^{k+1} + 6k + 6 - 10$

$= 4^{k+1} + 6k - 4$

$f(k+1) - f(k) = (4^{k+1} + 6k - 4) - (4^k + 6k - 10)$

$= (4(4^k) + 6k - 4) - (4^k + 6k - 10)$

$= 4(4^k) + \cancel{6k} - 4 - 4^k - \cancel{6k} + 10$

$= 3(4^k) + 6$

$= 3(4^k + 6k - 10) - 18k + 36$

$= 3f(k) + 18(2-k)$

$f(k+1) - f(k) = 3f(k) + 18(2-k)$

$f(k+1) = 4f(k) + 18(2-k)$

(4) Conclusion

If this result is true for $n=k$, then it is true for $n=k+1$. As the result has been shown to be true for $n=2$, then the result is true for all n.